

QUASI-IDEMPOTENT ROTA-BAXTER OPERATORS ARISING FROM QUASI-IDEMPOTENT ELEMENTS

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ABSTRACT. In this short note, we construct quasi-idempotent Rota-Baxter operators by quasi-idempotent elements and show that every finite dimensional Hopf algebra admits nontrivial Rota-Baxter algebra structures and tridendriform algebra structures. Several concrete examples are provided, including finite quantum groups and Iwahori-Hecke algebras.

1. INTRODUCTION

A Rota-Baxter algebra of weight $\lambda \in \mathbb{C}$ is an associative algebra R equipped with a linear endomorphism P , called a Rota-Baxter operator, verifying

$$P(a)P(b) = P(aP(b)) + P(P(a)b) + \lambda P(ab), \forall a, b \in R.$$

Sometimes we denote a Rota-Baxter algebra by the pair (R, P) or the triple (R, \cdot, P) if we need to emphasize the multiplication \cdot . The notion of Rota-Baxter algebra was first explicitly introduced by G.-C. Rota [25] based on the work of G. Baxter [5]. From then on, mathematicians began to study various aspects of these algebras and their applications (cf. [4], [14], [16], [9]). Besides their own interest in mathematics, Rota-Baxter algebras have also many important applications in mathematical physics. For example, they play an essential role in the Connes-Kreimer theory on the Hopf algebra approach to renormalization in perturbative quantum field theory (cf. [6], [7], [11], [12], [13]). They also appear in the study of Loday type algebras ([8], [10]), pre-Lie algebras ([3]), and pre-Poisson algebras ([1]).

Undoubtedly, good examples make a theory more dynamic. They provide intuitions and ideas for the further study of the subject. In the theory of Rota-Baxter algebras, there are many examples in some sense. Indeed, every algebra possesses at least a Rota-Baxter operator since the identity map is actually such an operator. However this is trivial and useless. For all these reasons, we would like to find systematic constructions of nontrivial Rota-Baxter algebras. Moreover, we would like to search interesting examples related to other branches of mathematics and mathematical physics, and then we can apply the tools from Rota-Baxter algebras to the study of these subjects. The recent article [18] is such an attempt. There, we use algebras in the category of Hopf modules to construct idempotent Rota-Baxter algebras. As an application, we obtain Rota-Baxter algebra structures on the positive part of a quantum group. On the other hand, the size of a Rota-Baxter algebra is usually very large. In other words, most of the known examples are infinite dimensional (cf. [15]). It is difficult to compute. The main purpose of this note is to give an explicit construction of finite dimensional Rota-Baxter algebras. Our

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starting point is the following easy observation. Let A be an associative algebra and a be an idempotent element of A , i.e., $a^2 = a$. We denote by l_a the operator of left multiplication by a . Then for any $b, c \in A$, we have

$$\begin{aligned}
 l_a(l_a(b)c) + l_a(bl_a(c)) - l_a(bc) &= a^2bc + abac - abc \\
 &= abc + abac - abc \\
 &= abac \\
 &= l_a(b)l_a(c).
 \end{aligned}$$

Therefore l_a is an idempotent Rota-Baxter operator of weight -1 . The problem is that idempotent elements do not always exist except the neutral element. In order to overcome this difficulty, we consider the so called quasi-idempotent elements. Recall that an element $a \in A$ is said to be quasi-idempotent if it satisfies $a^2 = k_a a$ for some $k_a \in \mathbb{C}$. This slight modification enables one to construct plenty of nontrivial examples having small size since such elements do exist for finite dimensional Hopf algebras. Note that finite dimensional Hopf algebras contain lots of significant examples, especially, Lusztig's small quantum groups ([22], [23]). This construction also contains other important examples such as Iwahori-Hecke algebras. So we can apply our construction to these examples and get interesting Rota-Baxter algebras.

This note is organized as follows. In Section 2, we recall the notion of quasi-idempotent element in an algebra and use it to construct quasi-idempotent Rota-Baxter operators. From this construction, we show that every finite dimensional Hopf algebra admits nontrivial Rota-Baxter algebra structures and tridendriform algebra structures. In Section 3, based on the constructions given in Section 2, we provide several interesting examples, including finite quantum groups and Iwahori-Hecke algebras.

2. CONSTRUCTIONS

For simplicity, we fix our ground field to be the complex number field \mathbb{C} throughout this note. All the objects we discuss are defined over \mathbb{C} unless otherwise specified.

We first recall the notion of quasi-idempotent operator which is introduced in [2].

Definition 2.1. Let A be an associative algebra and $\lambda \in \mathbb{C}$. A linear endomorphism ϕ of A is called a *quasi-idempotent operator of weight λ* if $\phi^2 = -\lambda\phi$. A nonzero element $\xi \in A$ is called a *quasi-idempotent element of weight λ* if $\xi^2 = -\lambda\xi$.

Now we use quasi-idempotent elements to construct quasi-idempotent Rota-Baxter operators.

Proposition 2.2. For a fixed quasi-idempotent element $\xi \in A$ of weight λ , we define $P_\xi : A \rightarrow A$ by $P_\xi(a) = \xi a$ for any $a \in A$. Then P_ξ is a quasi-idempotent Rota-Baxter operator of weight λ on A .

Proof. For any $a, b \in A$, we have

$$\begin{aligned}
 P_\xi(P_\xi(a)b) + P_\xi(aP_\xi(b)) + \lambda P_\xi(ab) &= \xi^2 ab + \xi a \xi b + \lambda \xi ab \\
 &= -\lambda \xi ab + \xi a \xi b + \lambda \xi ab
 \end{aligned}$$

$$\begin{aligned}
 &= \xi a \xi b \\
 &= P_\xi(a) P_\xi(b).
 \end{aligned}$$

□

Here the meaning of the weight of P_ξ is twofold: the weight of quasi-idempotent operators and the weight of Rota-Baxter operators.

Rota-Baxter algebras are closely related to tridendriform algebras.

Definition 2.3 ([21]). Let V be a vector space, and \prec, \succ and \cdot be three binary operations on V . The quadruple (V, \prec, \succ, \cdot) is called a *tridendriform algebra* if the following relations are satisfied: for any $x, y, z \in V$,

$$\begin{aligned}
 (x \prec y) \prec z &= x \prec (y * z), \\
 (x \succ y) \prec z &= x \succ (y \prec z), \\
 (x * y) \succ z &= x \succ (y \succ z), \\
 (x \succ y) \cdot z &= x \succ (y \cdot z), \\
 (x \prec y) \cdot z &= x \cdot (y \succ z), \\
 (x \cdot y) \prec z &= x \cdot (y \prec z), \\
 (x \cdot y) \cdot z &= x \cdot (y \cdot z),
 \end{aligned}$$

where $x * y = x \prec y + x \succ y + x \cdot y$.

Given a Rota-Baxter algebra (R, \cdot, P) of weight 1, we define $a \prec b = a \cdot P(b)$ and $a \succ b = P(a) \cdot b$ for $a, b \in R$. Then (R, \prec, \succ, \cdot) is a tridendriform algebra (see e.g. [8]).

Corollary 2.4. *Given a quasi-idempotent element ξ of weight $\lambda \neq 0$ in an algebra A , the operations \prec, \succ , and \cdot defined below endow a tridendriform algebra structure on A : for any $a, b \in A$,*

$$\begin{aligned}
 a \prec b &= \lambda^{-1} a \xi b, \\
 a \succ b &= \lambda^{-1} \xi a b, \\
 a \cdot b &= ab.
 \end{aligned}$$

If $\lambda = 0$, we can obtain a similar construction of dendriform algebra structures on A .

Notice that quasi-idempotent elements of weight -1 are just idempotent elements. So they are a generalization of idempotent elements. But as we mentioned in the introduction, quasi-idempotent elements always exist in finite dimensional Hopf algebras while idempotent elements do not. In order to provide quasi-idempotent elements, we first recall some standard notions and facts from Hopf algebras.

We always denote by $(H, \Delta, \varepsilon, S)$ a finite dimensional Hopf algebra with coproduct Δ , counit ε , and antipode S in the sequel. We adopt Sweedler's notion for coalgebra: $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ for any $h \in H$. We denote by H^* the linear dual of H , i.e., $H^* = \text{Hom}_{\mathbb{C}}(H, \mathbb{C})$, and by \langle, \rangle the usual contraction between H^* and

H . So $\langle a^*, a \rangle = a^*(a)$ for any $a^* \in H^*$ and $a \in H$. Then H^* is also a Hopf algebra with multiplication determined by $\langle a^*b^*, a \rangle = \sum \langle a^*, a_{(1)} \rangle \langle b^*, a_{(2)} \rangle$. Let l_{a^*} be the endomorphism of H^* defined by the left multiplication by a^* . Then there is a unique element x_H such that

$$\langle a^*, x_H \rangle = \text{Tr}(l_{a^*}), \quad \forall a^* \in H^*,$$

where Tr is the usual trace of endomorphisms.

The element x_H has the following properties (for a proof, one can see Proposition 10.7.6 in [24]).

Proposition 2.5. *Under the notation above, we have $\varepsilon(x_H) = \dim H$ and $x_H^2 = \varepsilon(x_H)x_H$.*

So x_H is a quasi-idempotent element of weight $-\dim H$. Furthermore, we have other quasi-idempotent elements in H . An element $\Lambda \in H$ is called a *left (resp. right) integral* for H if $a\Lambda = \varepsilon(a)\Lambda$ (resp. $\Lambda a = \varepsilon(a)\Lambda$) for all $a \in H$. Obviously, nonzero integrals are quasi-idempotent elements. It is well-known that the spaces of left integrals and right integrals are one-dimensional (see e.g. Theorem 10.2.2 in [24]). These two spaces do not coincide in general (see Example 3.2 below). And normally, the element x_H is not an integral since $cx_H = \varepsilon(c)x_H$ only for cocommutative element c (that means $\sum c_{(1)} \otimes c_{(2)} = \sum c_{(2)} \otimes c_{(1)}$).

By combining the discussions above, we see that P_{x_H} and P_Λ are Rota-Baxter operators on H . As a consequence, we have

Theorem 2.6. *Every finite dimensional Hopf algebra admits nontrivial Rota-Baxter algebra structures and tridendriform algebra structures.*

3. EXAMPLES

In this section, we exhibit our construction by four concrete examples. The first three are from Hopf algebras and quantum groups, and the last one is from Iwahori-Hecke algebras.

Example 3.1 (Group algebras). Let G be a finite group and $H = \mathbb{C}[G]$ its group algebra. Then H is a Hopf algebra with coproduct, counit, and antipode defined respectively by

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}, \quad \forall g \in G.$$

Both of the spaces of left integrals and right integrals of H are $\mathbb{C}\xi$ where $\xi = \sum_{g \in G} g$. Then the operator P_ξ on H defined by $P_\xi(h) = \varepsilon(h)\xi$ for $h \in H$ is a Rota-Baxter operator of weight $-|G|$.

Example 3.2 (Sweedler's four-dimensional Hopf algebra). Let H be the algebra generated by two elements x and y subject to

$$x^2 = \mathbf{1}, \quad y^2 = 0, \quad yx = -xy.$$

Then H is a four-dimensional algebra with a linear basis $\{\mathbf{1}, x, y, xy\}$ (see e.g. [19]). Moreover it is a Hopf algebra equipped with the following operations:

$$\begin{aligned} \Delta(x) &= x \otimes x, \quad \Delta(y) = \mathbf{1} \otimes y + y \otimes x, \\ \varepsilon(x) &= 1, \quad \varepsilon(y) = 0, \\ S(x) &= x, \quad S(y) = xy. \end{aligned}$$

We first compute the element x_H . Denote by $\{f_1, f_2, f_3, f_4\}$ the dual basis of $\{\mathbf{1}, x, y, xy\}$. The multiplication table of H^* is:

\cdot	f_1	f_2	f_3	f_4
f_1	f_1	0	f_3	0
f_2	0	f_2	0	f_4
f_3	0	f_3	0	0
f_4	f_4	0	0	0

Then we have

$$\mathrm{Tr}(l_{f_1}) = \mathrm{Tr}(l_{f_2}) = 2, \quad \mathrm{Tr}(l_{f_3}) = \mathrm{Tr}(l_{f_4}) = 0,$$

and hence $x_H = 2(\mathbf{1} + x)$. By the technique of comparison of coefficients, one can show that the spaces of left integrals and right integrals are $\mathbb{C}(y + xy)$ and $\mathbb{C}(y - xy)$ respectively. In general, quasi-idempotent elements of H are of the form $\xi = \mu_1(\mathbf{1} + x) + \mu_2 y + \mu_3 xy$ for some $\mu_1, \mu_2, \mu_3 \in \mathbb{C}$. Therefore, the operator P_ξ defined by the following actions is a Rota-Baxter operator of weight $-(2\mu_1 + \mu_2 + \mu_3)$:

$$\begin{aligned} P_\xi(\mathbf{1}) &= \mu_1(\mathbf{1} + x) + \mu_2 y + \mu_3 xy, \\ P_\xi(x) &= \mu_1(\mathbf{1} + x) - \mu_3 y - \mu_2 xy, \\ P_\xi(y) &= \mu_1(y + xy), \\ P_\xi(xy) &= \mu_1(y + xy). \end{aligned}$$

Example 3.3 (Lusztig's small quantum groups). Let \mathfrak{g} be a complex simple Lie algebra. In [22] and [23], Lusztig constructed some finite dimensional Hopf algebras attached to \mathfrak{g} which are called small quantum groups or Frobenius-Lusztig kernels nowadays. Numerous mathematicians have investigated these interesting Hopf algebras. Here we endow Rota-Baxter algebra structures on these small quantum groups. For simplifying statements and computations, we only give the explicit formulas for $\mathfrak{g} = \mathfrak{sl}(2)$. We follow the description of the small quantum group attached to $\mathfrak{sl}(2)$ given in [19]. Let d be a positive integer > 2 and q be a d -th primitive root of 1. Denote

$$e = \begin{cases} d, & \text{if } d \text{ is odd,} \\ d/2, & \text{if } d \text{ is even.} \end{cases}$$

We define \overline{U}_q to be the algebra generated by E, F, K, K^{-1} subject to

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1, \\ KE &= q^2 EK, \quad KF = q^{-2} FK, \\ EF - FE &= \frac{K - K^{-1}}{q - q^{-1}}, \\ E^e &= F^e = 0, \quad K^e = 1. \end{aligned}$$

Furthermore, \overline{U}_q is a Hopf algebra with coproduct, counit, and antipode defined below:

$$\begin{aligned}\Delta(E) &= 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \\ \Delta(K) &= K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}, \\ \varepsilon(E) &= \varepsilon(F) = 0, \quad \varepsilon(K) = \varepsilon(K^{-1}) = 1, \\ S(E) &= -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}, \quad S(K^{-1}) = K.\end{aligned}$$

As a vector space, \overline{U}_q is finite dimensional with a basis $\{E^i F^j K^l\}_{0 \leq i, j, l \leq e-1}$. One can verify directly that

$$\xi = E^{e-1} F^{e-1} (1 + K + K^2 + \cdots + K^{e-1})$$

is a left integral of \overline{U}_q . Then the operator P_ξ on \overline{U}_q of left multiplication by ξ is a Rota-Baxter operator of weight 0. We illustrate P_ξ by its action on the generators:

$$\begin{aligned}P_\xi(E) &= \sum_{n=0}^{e-1} \frac{q^{2n+e-2}}{q-q^{-1}} [e-1] E^{e-1} F^{e-2} K^{n+1} - \sum_{n=0}^{e-1} \frac{q^{2n+2-e}}{q-q^{-1}} [e-1] E^{e-1} F^{e-2} K^{n-1}, \\ P_\xi(F) &= 0, \\ P_\xi(K) &= \xi,\end{aligned}$$

where $[e-1] = \frac{q^{e-1}-q^{1-e}}{q-q^{-1}}$.

Example 3.4 (Iwahori-Hecke algebras). Let q be an indeterminate and \mathcal{A} be the ring $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ of Laurent polynomials in $q^{\frac{1}{2}}$. Let (W, S) be a Coxeter system and \mathcal{H} be the corresponding Iwahori-Hecke algebras over \mathcal{A} . Then as an \mathcal{A} -module \mathcal{H} has a standard basis $\{T_w | w \in W\}$, and the multiplication relations for this basis are:

$$\begin{aligned}T_w T_{w'} &= T_{ww'}, \quad \text{if } l(ww') = l(w) + l(w'), \\ T_s^2 &= (q-1)T_s + q, \quad \text{for } s \in S,\end{aligned}$$

where l is the usual length function on W . For any $s \in S$ we denote $C_s = q^{-\frac{1}{2}}(T_s - q)$. They are parts of the Kazhdan-Lusztig basis of \mathcal{H} (see [20] and [17]). Then we have

$$\begin{aligned}C_s^2 &= (q^{-\frac{1}{2}}(T_s - q))^2 \\ &= q^{-1}(T_s^2 - 2qT_s + q^2) \\ &= q^{-1}((q-1)T_s + q - 2qT_s + q^2) \\ &= -(q^{\frac{1}{2}} + q^{-\frac{1}{2}})q^{-\frac{1}{2}}(T_s - q) \\ &= -(q^{\frac{1}{2}} + q^{-\frac{1}{2}})C_s.\end{aligned}$$

So C_s is an quasi-idempotent element of weight $q^{\frac{1}{2}} + q^{-\frac{1}{2}}$, and hence the operator P_{C_s} given below is a Rota-Baxter operator of weight $q^{\frac{1}{2}} + q^{-\frac{1}{2}}$ on \mathcal{H} :

$$P_{C_s}(T_w) = \begin{cases} q^{-\frac{1}{2}}T_{sw} - q^{\frac{1}{2}}T_w, & \text{if } l(sw) > l(w), \\ q^{\frac{1}{2}}T_{sw} - q^{-\frac{1}{2}}T_w, & \text{if } l(sw) < l(w). \end{cases}$$

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